GROUP ACTIONS, CYCLIC COVERINGS AND FAMILIES OF K3-SURFACES

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ABSTRACT. In this paper we describe six pencils of $K3$-surfaces which have large Picard-number ($15 \leq \rho \leq 20$) and contain precisely five singular fibers: four have A-D-E singularities and one is non-reduced. In particular we describe these surfaces as cyclic coverings of the $K3$-surfaces of [BS]. In many cases using this description and lattice-theory we are able to compute the exact Picard-number and to describe explicitly the Picard-lattices.

0. INTRODUCTION

In the last years using different methods (toric geometry, mirror symmetry, etc.) have been constructed and studied many families of $K3$-surfaces with large Picard-Number and small number of singular fibers (see e.g. [D], [VY] and [Be]). In these notes using group actions and cyclic coverings we describe some new one. In [BS] the authors describe three pencils of $K3$-surfaces where the generic surface has Picard-number 19 and the pencils contain four singular fibers with singularities of $A-D-E$ type and one non-reduced fiber. The families arise as minimal resolutions of quotients X_λ/G were G is a subgroup of $SO(4)$ containing the Heisenberg group and $\{X_\lambda\}_{\lambda \in \mathbb{P}_1}$ is a G-invariant pencil of surfaces in $\mathbb{P}_3$, the latter are described in [S1]. The groups G are the so-called *bi-polyhedral groups*: $T \times T$, $O \times O$, $I \times I$, where T, O and I denote the tetrahedral group, the octahedral group and the icosahedral group in $SO(3)$ (we recall their precise definition in the first Section), the pencils X_λ^6, X_λ^8, X_λ^{12} of $T \times T-$, $O \times O-$, resp. $I \times I-$invariant surfaces, have degrees 6, 8, resp. 12. In these notes we consider some special normal subgroup H of $T \times T$ and $O \times O$ ($I \times I$ is simple) so that the minimal resolutions of the quotients are pencils of $K3$-surfaces, these have then five singular fibers (one is non-reduced) and large Picard-number. In Section 2 we describe all the subgroups H with this property. In the Sections 3, 4, 5 and 6 we describe six new one-dimensional families of $K3$-surfaces: the generic surface of two of the pencils has Picard-number $\rho = 19$ and there are four singular surfaces with $\rho = 20$ as in the case of the families $X_\lambda^6/T \times T$ or $X_\lambda^8/O \times O$ of [BS]. The other four pencils seem to have smaller Picard-number. We give in any case a lower bound for ρ and in each case except one we could identify surfaces with $\rho = 20$. The methods which we use in these Sections are essentially the same as in [BS]. In the Sections 7 and 9, we describe the K3-surfaces as cyclic coverings of the K3-surfaces of [BS], more precisely let $Y_{\lambda,T \times T}$ and $Y_{\lambda,O \times O}$ denote the minimal resolutions of the quotients $X_\lambda^6/T \times T$ and $X_\lambda^8/O \times O$, then taking two special 3-divisible classes of rational curves in the Neron-Severi group $NS(Y_{\lambda,T \times T})$ and two special 2-divisible classes of rational curves in the Neron-Severi group $NS(Y_{\lambda,O \times O})$ one can construct cyclic coverings (two for each surface) which correspond (up to contract some (-1)-curves) to the minimal resolutions of the surfaces X_λ^6/H and X_λ^8/H' for some normal subgroup H of $T \times T$ and H' of $O \times O$. By doing cyclic coverings of the latter,

one obtains more $K3$-surfaces. In Section 8 and 9 by using these descriptions and the results of [BS], Section 6, we compute explicitly the Picard-lattice of the $K3$-surfaces of two of the families, more precisely of those where the generic surface has $\rho = 19$ and of the four special surfaces with $\rho = 20$ contained in these families. We compute also the Picard-lattice of the surfaces with $\rho = 20$ in the other pencils.

I would like to thank W. Barth for introducing me to cyclic coverings and for many useful discussions.

1. NOTATIONS AND PRELIMINARIES

There are two classical $2 : 1$ coverings:

$$\rho : SU(2) \to SO(3) \text{ and } \sigma : SU(2) \times SU(2) \to SO(4).$$

Let $G_i \subset SO(3)$, $i = 1, 2$ denote the polyhedral group T or O. We consider the binary group $\tilde{G}_i := \rho^{-1}(G_i) \subset SU(2)$ and the σ-image: $\sigma(\tilde{G}_1 \times \tilde{G}_2) \subset SO(4)$, which by abuse of notation we denote with $G_1 \times G_2$ and we denote an element of $SU(2) \times SU(2)$ and its image in $SO(4)$ by (p_1, p_2). These groups have been studied in [S1], there the group $T \times T$ is called G_6 and the group $O \times O$ is called G_8. We will denote by $X_\lambda^6 = s_6 + \lambda q^3$ and by $X_\lambda^8 = s_8 + \lambda q^4$ the pencils of $T \times T$- and of $O \times O$-invariant surfaces in $\mathbb{P}_3$, which are described in [S1], s_6 denotes a $T \times T$-invariant homogeneous polynomial of degree six and s_8 denotes an $O \times O$-invariant homogeneous polynomial of degree eight, $q := x_0^2 + x_1^2 + x_2^2 + x_3^2$ is the equation of the quadric $\mathbb{P}_1 \times \mathbb{P}_1$ in $\mathbb{P}_3$. The base locus of the pencils X_λ^n are $2n$ lines $(n = 6, 8)$ on the quadric, n in each ruling. These pencils contain exactly four nodal surfaces. We recall the value of the λ_i and the number of nodes on X_{λ_i} in the table below (cf. [S1]):

	$n = 6$			$n = 8$			
λ_1	λ_2	λ_3	λ_4	λ_1	λ_2	λ_3	λ_4
-1	$-\frac{2}{3}$	$-\frac{7}{12}$	$-\frac{1}{4}$	-1	$-\frac{3}{4}$	$-\frac{9}{16}$	$-\frac{5}{9}$
12	48	48	12	24	72	144	96

We recall also the matrix:

$$C := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \in O(4),$$

which operates on an element $(p_1, p_2) \in G_1 \times G_2$ by:

$$C^{-1}(p_1, p_2)C = (p_2, p_1).$$

Moreover we specify the following matrices of $SO(4)$:

$$(q_1, 1) = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad (1, q_1) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$(q_2, 1) = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad (1, q_2) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

$$(p_3, 1) = \tfrac{1}{2} \begin{pmatrix} 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \quad (1, p_3) = \tfrac{1}{2} \begin{pmatrix} 1 & 1 & -1 & 1 \\ -1 & 1 & -1 & -1 \\ 1 & 1 & 1 & -1 \\ -1 & 1 & 1 & 1 \end{pmatrix},$$

$$(p_4, 1) = \tfrac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \quad (1, p_4) = \tfrac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix},$$

Using these matrices the groups have the following generators:

Group	Generators
$T \times T$	$(q_2, 1), (1, q_2), (p_3, 1), (1, p_3)$
$O \times O$	$(q_2, 1), (1, q_2), (p_3, 1), (1, p_3), (p_4, 1), (1, p_4)$

Denote by PG the image of a subgroup $G \subset SO(4)$ in $\mathbb{P}GL(4)$. We define the *types* of lines in $\mathbb{P}_3$ which are fixed by elements $(p_1, p_2) \in PG$ of order $2, 3$ or 4 in the following way:

order	2	3	4
type	M	N	R

2. NORMAL SUBGROUPS

In [S2] the author classifies all the subgroups of $SO(4)$ which contain the Heisenberg group $V \times V$. Here we consider all the normal subgroups of $T \times T$ and $O \times O$ which contain the subgroup $V \times V$ and $T \times T$. The group $I \times I$ does not contain normal subgroups. We denote by H such a normal subgroup, by $o(H)$ its order and by $i(H) = [G : H]$ the index of H in G ($G = T \times T$ or $O \times O$). We list below all the groups H and their generators, following the notation of [S2]. Moreover we do not consider separately the groups H and $C^{-1}HC$ or, in general, groups which are conjugate in $O(4)$.

H	generators	$o(H)$	G	$i(H)$
$T \times V$	$(q_1,1),(1,q_1)$ $(p_3,1),(1,p_3)$	96	$T \times T$	3
$(TT)'$	$(q_1,1),(1,q_1)$ $(q_2,1),(1,q_2)$ (p_3,p_3)	96	$T \times T$	3
$V \times V$	$(q_1,1),(1,q_1)$ $(q_2,1),(1,q_2)$	32	$T \times T$	9
$O \times T$	$(q_1,1),(1,q_1)$ $(p_3,1),(1,p_3)$ $(p_4,1)$	576	$O \times O$	2
$(OO)''$	$(q_1,1),(1,q_1)$ $(p_3,1),(1,p_3)$ (p_4q_2,p_4q_2)	576	$O \times O$	2
$T \times T$	$(q_1,1),(1,q_1)$ $(p_3,1),(1,p_3)$	288	$O \times O$	4

3. Fix-points

We analyze the different kind of fix-points for elements of the subgroups $PH \subset PG$ in the same way as in [BS]. Recall that the elements of the form $(p,1)$ or $(1,p)$ have each two disjoint lines of fix points contained in one ruling, respectively in the other ruling of the quadric (cf. [S1], 5.4 p. 439).

1) *Fix-points on the quadric.* The subgroups $G_1 \times 1$ and $1 \times G_2 \subset PH$ operate on the two rulings of the quadric and determine orbits of lines, which are also PH-orbits of lines. We resume the lengths of the orbits in the following tables. In the first row we write the order of the element which fixes two lines of the orbit:

order of $(p,1)$	2	3	4
$T \times V$	6	4,4	–
$O \times T$	12	8	6
$(TT)'$	6	–	–
$(OO)''$	6	8	–
$V \times V$	2,2,2	–	–
$T \times T$	6	4,4	–

order of $(1,p)$	2	3	4
$T \times V$	2,2,2	–	–
$O \times T$	6	4,4	–
$(TT)'$	6	–	–
$(OO)''$	6	8	–
$V \times V$	2,2,2	–	–
$T \times T$	6	4,4	–

In particular observe that in the case of the groups $(TT)'$ and $(OO)''$ the meeting points of the fix-lines of the two rulings of $\mathbb{P}_1 \times \mathbb{P}_1$ split into three orbits of length 12 and two orbits of length 32, in the other cases these meeting-points form just one orbit.

2) *Fix-points off the quadric.* We denote by F_L the fix-group of a line L of $\mathbb{P}_3$ in PH and by H_L the stabilizer group of L in PH, i.e.

$$F_L := \{h \in PH \ s.t. \ hx = x \ for \ all \ x \in L\}$$
$$H_L := \{h \in PH \ s.t. \ hL = L\}.$$

Moreover denote by $\ell(L)$ the length of the G-orbit of the line L and by g a representative of a conjugacy class in G:

group	$T \times V$			$(TT)'$				$V \times V$
g	(q_1, q_1)	(q_1, q_2)	(q_1, q_3)	(q_1, q_1)	(q_1, q_2)	(q_1, q_3)	(p_3, p_3)	(q_i, q_j)
F_L	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_2$
type	M_1	M_2	M_3	M_1	M_2	M_3	N	M_{ij}
$\ell(L)$	6	6	6	6	6	6	16	2
$\|H_L\|/\|F_L\|$	4	4	4	4	4	4	1	4

Here we denote by q_3 the product of q_1 and q_2. In the last column of the table the sum runs over $i, j = 1, 2, 3$. In this case we have nine distinct conjugacy classes with just one element.

group	$O \times T$			$(OO)''$				$T \times T$		
g	(q_1, q_1)	(p_3, p_3)	$(p_4 q_2, q_2)$	(p_4, p_4)	(p_3, p_3)	(p_3^2, p_3)	$(p_4 q_2, p_4 q_2)$	(q_2, q_2)	(p_3, p_3)	(p_3^2, p_3)
F_L	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_2$	$\mathbb{Z}_4$	$\mathbb{Z}_3$	$\mathbb{Z}_3$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_3$
type	M	N	M'	R	N	N'	M	M	N	N'
$\ell(L)$	18	32	36	18	16	16	72	18	16	16
$\|H_L\|/\|F_L\|$	8	3	3	4	8	8	2	4	3	3

Remark 3.1. *By taking the generator (p_3^2, p_3) for $(TT)'$ instead of (p_3, p_3) we find a group $(TT)''$ which is conjugate in $O(4)$ to $(TT)'$. The singularities in the quotient $X_\lambda^6/(TT)''$ are the same as those on $X_\lambda^6/(TT)'$.*

4. Quotient singularities

We consider now the projections:

$$\pi_H : X_\lambda^6 \longrightarrow X_\lambda^6/H, \quad \pi_{H'} : X_\lambda^8 \longrightarrow X_\lambda^8/H'$$

with $H = T \times V$, $(TT)'$ or $V \times V$; $H' = O \times T$, $(OO)''$ or $T \times T$. In this Section we run the same program as in [BS], Section 3 and describe the singularities of the quotients (for the details cf. [BS]) .

1) *Fix-lines on q.* The image in the quotient of the lines of the base locus of the pencils X_λ^6 and X_λ^8 and of the intersection points of the lines of the base locus are smooth. Observe that the points of intersection of the lines of the base locus of the pencils form one orbit under the action of $T \times V$, $V \times V$, $O \times T$ and $T \times T$. In the case of the groups $(TT)'$ we have three orbits and in the case of the group $(OO)''$ we have two orbits, as described in Section 3, this means that the lines in the quotient will meet three times and two times. Now we consider the points of intersection of the lines of the base locus with the other fix-lines on q. In the table below we do not write the groups $(TT)'$ and $V \times V$ because they do not have other fix-points on q other then the lines of the base locus. We denote by Fix(P) the fix-group in PG of a point P. In the next table we write the length and the number of orbits of fix-points, and we describe which kind of singularities do we have in the quotient:

group	$T \times V$	$O \times T$			$(OO)''$		$T \times T$	
Fix(P)	$\mathbb{Z}_3 \times \mathbb{Z}_2$	$\mathbb{Z}_3 \times Z_2$	$\mathbb{Z}_4 \times \mathbb{Z}_3$	$\mathbb{Z}_2 \times \mathbb{Z}_3$	$\mathbb{Z}_2 \times \mathbb{Z}_3$	$\mathbb{Z}_3 \times \mathbb{Z}_2$	$\mathbb{Z}_3 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_3$
length	8	48	24	48	48	48	24	24
number	6	1	2	2	1	1	2	2
sing.	$6A_2$	$1A_1$	$2A_3$	$2A_1$	$1A_1$	$1A_1$	$2A_1$	$2A_1$

2)*Fix-lines off q.* Denote by $o(L)$ the order of the fix-group F_L of L. The number of points not on q cut out on X_λ by L is:

group	$T \times V$	$(TT)'$		$V \times V$	$O \times T$		$(OO)''$		$T \times T$	
$o(L)$	2	2	3	2	2	3	4	3	2	3
number	4	4	6	4	8	6	8	6	8	6

In the next table we show in each case length and number of H_L-orbits, the number and type(s) of the quotient singularity(ies):

group	$T \times V$			$(TT)'$				$V \times V$
$o(L)$	2	2	2	2	2	2	3	2
type	M_1	M_2	M_3	M_1	M_2	M_3	N	M_{ij}
length	4	4	4	4	4	4	1	4
number	1	1	1	1	1	1	6	1
singularities	A_1	A_1	A_1	A_1	A_1	A_1	$6A_2$	A_1

group	$O \times T$			$(OO)''$				$T \times T$		
$o(L)$	2	3	2	4	3	3	2	2	3	3
type	M	N	M'	R	N	N'	M	M	N	N'
length	8	3	3	4	6	6	2	4	3	3
number	1	2	2	2	1	1	4	2	2	2
singularities	A_1	$2A_2$	$2A_1$	$2A_3$	A_2	A_2	$4A_1$	$2A_1$	$2A_2$	$2A_2$

3) *The singular surfaces.* We denote by ns the number of nodes on the surfaces and by F the fix-group of a node in H. In the table below, we give the number of orbits of nodes and their fix-groups in PH, PH' and we describe the singularities in the quotient. For doing this recall the Proposition 3.1 of [BS]:

Proposition 4.1. *Let X be a nodal surface with $F \subset SO(3)$ the fix-group of the node. Then the image of this node on X/H is a quotient singularity locally isomorphic with $\mathbb{C}^2/\tilde{F}$, where $\tilde{F} \subset SU(2)$ is the binary group which corresponds to F.*

group	$T \times V$				$(TT)'$				$V \times V$			
λ	λ_1	λ_2	λ_3	λ_4	λ_1	λ_2	λ_3	λ_4	λ_1	λ_2	λ_3	λ_4
ns	12	48	48	12	12	48	48	12	12	48	48	12
orbit	1	1	1	1	3	3	1	1	3	3	3	3
F	$\mathbb{Z}_2 \times \mathbb{Z}_2$	id	id	$\mathbb{Z}_2 \times \mathbb{Z}_2$	T	$\mathbb{Z}_3$	id	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	id	id	$\mathbb{Z}_2 \times \mathbb{Z}_2$
lines	$1M_1$	–	–	$1M_1$	$3M_i$	$1N$	–	$1M_1$	$3M_{ij}$	–	–	$3M_{ij}$
meeting	$1M_2$			$1M_2$	$4N$			$1M_2$				
	$1M_3$			$1M_3$				$1M_3$				
sing.	D_4	A_1	A_1	D_4	$3E_6$	$3A_5$	A_1	D_4	$3D_4$	$3A_1$	$3A_1$	$3D_4$

group	$O \times T$				$(OO)''$				$T \times T$			
λ	λ_1	λ_2	λ_3	λ_4	λ_1	λ_2	λ_3	λ_4	λ_1	λ_2	λ_3	λ_4
ns	24	72	144	96	24	72	144	96	24	72	144	96
orbit	1	1	1	1	2	1	1	2	2	1	1	2
F	T	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	O	$\mathbb{Z}_4$	$\mathbb{Z}_2$	D_3	T	$\mathbb{Z}_2$	id	$\mathbb{Z}_3$
lines	$3M$	$1M$	$1M'$	$1N$	$3R$	$1R$	$1M$	$1N(N')$	$3M$	$1M$	–	$1N(N')$
meeting	$4N$	$2M'$			$4N(N')$			$3M$	$4N(N')$			
					$6M$							
sing.	E_6	D_4	A_3	A_5	$2E_7$	A_7	A_3	$2D_5$	$2E_6$	A_3	A_1	$2A_5$

5. Rational curves

Let

$$\mu : Y_{\lambda,H} \longrightarrow X_\lambda/H$$

be the minimal resolution of the singularities of X_λ/H, which denotes one of the six quotients of Section 4. In the following table we give the number of rational curves coming from the curves of the base locus of X_λ (denote it by ν_1) and from the resolution of the singularities. The latter are of three kinds: those coming from the intersection points of the lines of the base locus with other fix-lines on q, those coming from fix-points which are off q and do not come from nodes of X_λ, and those coming from the nodes. We denote their numbers by ν_2, ν_3 and ν_4, then the total number of rational curves is $\nu := \nu_1 + \nu_2 + \nu_3 + \nu_4$. We give the discriminant, d, of the intersection matrix too.

1. *The smooth X_λ.*

group	$T \times V$	$(TT)'$	$V \times V$	$O \times T$	$(OO)''$	$T \times T$
ν_1	4	2	6	3	2	4
ν_2	12	–	–	9	2	4
ν_3	3	15	9	7	14	10
ν	19	17	15	19	18	18
d	$2^5 \cdot 3^3 \cdot 5$	$2^3 \cdot 3^6 \cdot 5$	$2^{13} \cdot 5$	$2^5 \cdot 3^3 \cdot 7$	$-2^8 \cdot 3^2 \cdot 7$	$-2^2 \cdot 3^6 \cdot 7$

2. *The singular X_λ.* In this case the surfaces X_λ dont have extra singularities on q, hence the number ν_1 and ν_2 remain the same as above and we do not write them again.

group	$T \times V$				$(TT)'$			
λ	λ_1	λ_2	λ_3	λ_4	λ_1	λ_2	λ_3	λ_4
ν_3	–	3	3	–	–	3	15	12
ν_4	4	1	1	4	18	15	1	4
ν	20	20	20	20	20	20	18	18
d	$-2^4 \cdot 3^3 \cdot 5$	$-2^6 \cdot 3^3 \cdot 5$	$-2^6 \cdot 3^3 \cdot 5$	$-2^4 \cdot 3^3 \cdot 5$	$-3^3 \cdot 5$	$-2^6 \cdot 3^3 \cdot 5$	$-2^4 \cdot 3^6 \cdot 5$	$-2^2 \cdot 3^6 \cdot 5$

group	$V \times V$				$O \times T$			
λ	λ_1	λ_2	λ_3	λ_4	λ_1	λ_2	λ_3	λ_4
ν_3	–	9	9	–	2	4	5	3
ν_4	12	3	3	12	6	4	3	5
ν	18	18	18	18	20	20	20	20
d	$-2^{10} \cdot 5$	$-2^{16} \cdot 5$	$-2^{16} \cdot 5$	$-2^{10} \cdot 5$	$-2^4 \cdot 3^2 \cdot 7$	$-2^4 \cdot 3^3 \cdot 7$	$-2^5 \cdot 3^3 \cdot 7$	$-2^6 \cdot 3^2 \cdot 7$

group	$(OO)''$				$T \times T$			
λ	λ_1	λ_2	λ_3	λ_4	λ_1	λ_2	λ_3	λ_4
ν_3	–	8	12	6	–	8	10	2
ν_4	16	7	3	10	12	3	1	10
ν	20	19	19	20	20	19	19	20
d	$-2^4 \cdot 7$	$2^7 \cdot 3^2 \cdot 7$	$2^8 \cdot 3^2 \cdot 7$	$-2^8 \cdot 7$	$-3^4 \cdot 7$	$2^2 \cdot 3^6 \cdot 7$	$2^3 \cdot 3^6 \cdot 7$	$-2^4 \cdot 3^4 \cdot 7$

6. K3-surfaces

Since the groups H and H' contain the subgroups $V \times V$ of $T \times T$ resp. $T \times T$ of $O \times O$ the projections π_H and $\pi_{H'}$ of Section 4 are ramified on the lines of the base locus of the families X_λ with ramification index two, and three. A computation as in [BS] Section

5, shows that the quotients are $K3$-surfaces with Picard-number at least the number of rational curves given in Section 5.

Proposition 6.1. *The structure of the K3-surfaces* $Y_{\lambda,T\times V}$, $Y_{\lambda,O\times T}$ *varies with* λ.

Proof. [BS] Proposition 5.2 and Lemma 5.2. □

Corollary 6.1. *The general K3-surface* $Y_{\lambda,T\times V}$, $Y_{\lambda,O\times T}$ *has Picard-number 19.*

For the other pencils, and in the special cases too, we are not always able to identify exactly the Picard-number, but we can give a lower bound for it, which is the number ν of rational curves given in the tables above.
We are now going to describe another construction for these pencils of $K3$-surfaces.

7. Cyclic Coverings

Consider now the pairs G and H so that G/H is cyclic, in our cases either $|G/H| = 3$ or $|G/H| = 2$ (cf. Section 2), and consider the map:

$$\pi : X_\lambda/H \longrightarrow X_\lambda/G$$

7.1. **The general case.** For the moment assume that X_λ is smooth. The group G/H acts on the points of the fiber $\pi^{-1}(P)$. If the point P is not fixed by G/H then the map is $3:1$ or $2:1$ there. If P is fixed by G/H then we have a singularity on X_λ/H, more precisely an A_2 or an A_1, now the fiber $\pi^{-1}(P)$ is one point and the map has multiplicity 2 or 3 there (cf. [M2] Lemma 3.6 p. 80). Consider now the minimal resolutions Y_G and Y_H of the singular points of X_λ/G and X_λ/H. As shown in [BS] and in Section 6 these surfaces are K3 and we have a rational map:

$$\gamma : Y_H - -- \rightarrow Y_G$$

which is $3:1$ or $2:1$. Observe that this map is not defined over the (-2)-rational curves in the blow up of the singular points of X_λ/G which comes from fix-points of G/H on X_λ/H.
In this Section we describe the map γ in another way, more precisely by using cyclic coverings. For the general theory about 2-cyclic coverings and 3-cyclic coverings we send back to the article [N] of Nikulin and to the articles [M1] of Miranda and [T] of Tan. For the convenience of the reader in Figure 1. we recall the configurations of (-2)-rational curves on the smooth surfaces $Y_{T\times T}$ and on $Y_{O\times O}$ given in [BS].
By [BS], Proposition 6.1, we have the following 3-divisible classes in $NS(Y_{T\times T})$:

$$\mathcal{L} := L_1 - L_2 + L_4 - L_5 + N_1 - N_2 + N_3 - N_4 + N_5 - N_6 + N_7 - N_8,$$
$$\mathcal{L}' := L_1' - L_2' + L_4' - L_5' + N_1 - N_2 + N_3 - N_4 - N_5 + N_6 - N_7 + N_8,$$

and also:

$$\mathcal{L} - \mathcal{L}' = L_1 - L_2 + L_4 - L_5 - L_1' + L_2' - L_4' + L_5' + 2(N_5 - N_6 + N_7 - N_8),$$
$$\mathcal{L} + \mathcal{L}' = L_1 - L_2 + L_4 - L_5 + L_1' - L_2' + L_4' - L_5' + 2(N_1 - N_2 + N_3 - N_4).$$

Making reduction modulo three we find the classes:

$$\mathcal{M} := L_1 - L_2 + L_4 - L_5 - L_1' + L_2' - L_4' + L_5' - (N_5 - N_6 + N_7 - N_8),$$
$$\mathcal{M}' := L_1 - L_2 + L_4 - L_5 + L_1' - L_2' + L_4' - L_5' - (N_1 - N_2 + N_3 - N_4).$$

$Y_{T \times T}$
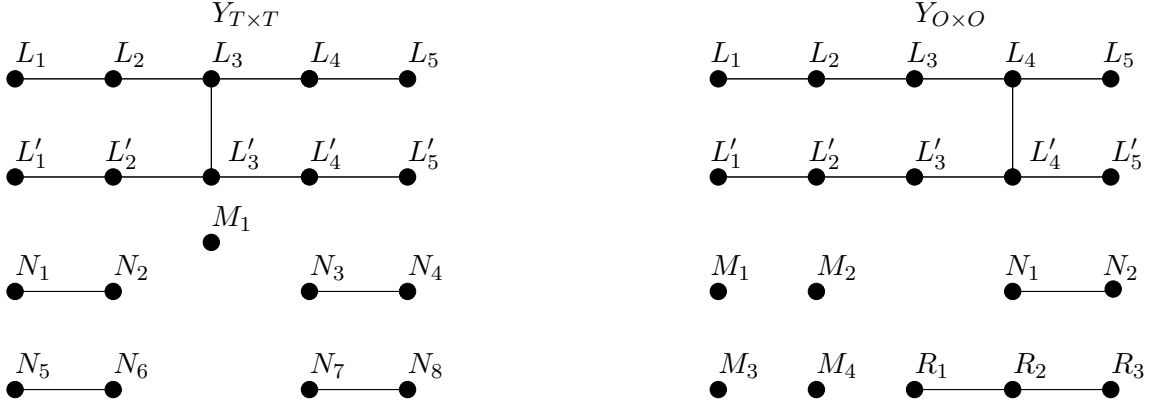
$L_1 \quad L_2 \quad L_3 \quad L_4 \quad L_5$

$L'_1 \quad L'_2 \quad L'_3 \quad L'_4 \quad L'_5$

M_1

$N_1 \quad N_2 \qquad N_3 \quad N_4$

$N_5 \quad N_6 \qquad N_7 \quad N_8$

$Y_{O \times O}$

$L_1 \quad L_2 \quad L_3 \quad L_4 \quad L_5$

$L'_1 \quad L'_2 \quad L'_3 \quad L'_4 \quad L'_5$

$M_1 \quad M_2 \qquad N_1 \quad N_2$

$M_3 \quad M_4 \qquad R_1 \quad R_2 \quad R_3$

Fig. 1

In $NS(Y_{O \times O})$ we have the following 2-divisible classes:

$$\mathcal{L} := L_1 + L_3 + L_5 + M_1 + M_3 + M_4 + R_1 + R_3,$$
$$\mathcal{L}' := L'_1 + L'_3 + L'_5 + M_2 + M_3 + M_4 + R_1 + R_3.$$

Consider also the classes $\mathcal{L} + \mathcal{L}'$ and $\mathcal{L} - \mathcal{L}'$, which after reduction modulo 2 are the same as:

$$\mathcal{M} := L_1 + L_3 + L_5 + L'_1 + L'_3 + L'_5 + M_1 + M_2.$$

These classes consist of six disjoint A_2-configurations of curves and of eight disjoint A_1-configurations of curves (in accord to [T] and [N]). These are the resolutions of A_2 and A_1 singularities of X_λ/G which arise by doing the quotient of X_λ/H by G/H. We construct the 3-cyclic coverings and the 2-cyclic coverings by using the divisors $\mathcal{L}, \mathcal{L}', \mathcal{M}, \mathcal{M}'$. To avoid to produce singularities by doing this, we first blow up the meeting points of the A_2-configurations. Call $Y^0_{T \times T}$ the surface which we obtain after these blow-ups. The meeting points are replaced by (-1)-curves and the two (-2)-curves become now (-3)-curves. Denote by $\phi : Y^1_{T \times T} \longrightarrow Y^0_{T \times T}$ the 3-cyclic covering with branching divisor $\mathcal{L}, \mathcal{L}'$ or $\mathcal{M}$, $\mathcal{M}'$:

Proposition 7.1. *A configuration of curves on $Y^0_{T \times T}$:*

$$-3 \qquad -1 \qquad -3$$
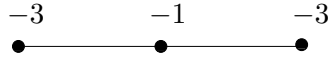
becomes a configuration:

$$-1 \qquad -3 \qquad -1$$

on $Y^1_{T \times T}$.
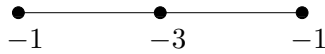
Proof. We do the computation for one configuration of curves $L_1 - L_2$, this is the same in the other cases. Denote again by L_1 and L_2 the curves on $Y^0_{T \times T}$ which now are (-3)-curves

$$
\begin{array}{ccc}
Y_G^1 & \xrightarrow{\ \phi\ } & Y_G^0 \\
\swarrow & & \searrow \\
Y_H & -\xrightarrow{\ \gamma\ }- & Y_G \\
\searrow & & \swarrow \\
X_\lambda/H & \xrightarrow{\ \pi\ } & X_\lambda/G
\end{array}
$$

Fig. 2

and denote by E the exceptional (-1)-curve. By the properties of cyclic coverings we have $\phi^* L_i = 3\tilde{L}_i$. Where $\tilde{L}_i$ is the strict transform of L_i. Then:

$$9(\tilde{L}_i)^2 = (\phi^* L_i)^2 = (\deg\phi)L_i^2 = -9.$$

Hence $(\tilde{L}_i)^2 = -1$. Since $E\cdot(L_1 - L_2) = 0$ the map ϕ is not ramified on E and the restriction $\phi_{|\tilde{E}}$ is $3:1$ onto E. Hence we have $\phi^* E = \tilde{E}$ and $\tilde{E}^2 = (\phi^* E)^2 = (\deg\phi)E^2 = 3E^2 = -3$.
$\square$

Our surface $Y_{T\times T}^1$ is now no more minimal. By blowing down the (-1)-curves, the curve $\tilde{E}$ becomes also a (-1)-curve so we blow it down too. We call the new smooth surfaces $T_{\mathcal{L}}$, $T_{\mathcal{L}'}$ resp. $T_{\mathcal{M}}$, $T_{\mathcal{M}'}$.

On $Y_{O\times O}$ we construct directly the 2-cyclic coverings with branching divisors $\mathcal{L}$ and $\mathcal{M}$, because these consist of disjoint smooth rational curves. A computation as in Proposition 7.1 shows that on the coverings the (-2)-curves of the branching divisors became (-1)-curves, hence we can blow them down. We call the surfaces which we obtain $O_{\mathcal{L}}$, $O_{\mathcal{L}'}$ and $O_{\mathcal{M}}$.

By Proposition 7.2 below and Remark 7.1 follows that the surfaces $T_{\mathcal{L}}$, $T_{\mathcal{L}'}$, $T_{\mathcal{M}}$ and $O_{\mathcal{L}}$, $O_{\mathcal{L}'}$, $O_{\mathcal{M}}$ are minimal K3-surfaces and by construction are exactly the surfaces Y_H which are obtained as the minimal resolutions of X_λ/H. We have a commutative diagram as in Figure 2.

In the diagram in the case of $G = O \times O$, we have $Y_G^0 = Y_G$ and the map to Y_G is the identity. This construction allow us to describe in another way the (-2)-curves on Y_H which in some cases determine the rank of the Neron-Severi group of the $K3$-surfaces.

Proposition 7.2. *The surfaces $T_{\mathcal{L}}$, $T_{\mathcal{M}}$, $O_{\mathcal{L}}$, $O_{\mathcal{M}}$ are K3, containing the following numbers of independent rational (-2) curves coming from the rational curves of $Y_{T\times T}$ and $Y_{O\times O}$ of Figure 1:*

$T_{\mathcal{L}}$	$T_{\mathcal{M}}$	$O_{\mathcal{L}}$	$O_{\mathcal{M}}$
19	17	19	18

Moreover we have the following isomorphism of surfaces:

$T_{\mathcal{L}}$	$T_{\mathcal{M}}$	$O_{\mathcal{L}}$	$O_{\mathcal{M}}$
$Y_{T\times V}$	$Y_{(TT)'}$	$Y_{O\times T}$	$Y_{(OO)''}$

We have these isomorphisms up to conjugate with C the group $T \times V$, up to change (p_3, p_3) with (p_3^2, p_3) in $(TT)'$ and up to conjugate with C the group $O \times T$.

Proof. 1. Observe that the curves $L_1, L_2, L_4, L_5, N_1, N_2, N_3, N_4, N_5, N_6, N_7, N_8$ "disappear" on $T_{\mathcal{L}}$, as the curves $L_1, L_2, L_4, L_5, L_1', L_2', L_4', L_5', N_5, N_6, N_7, N_8$ "disappear" on

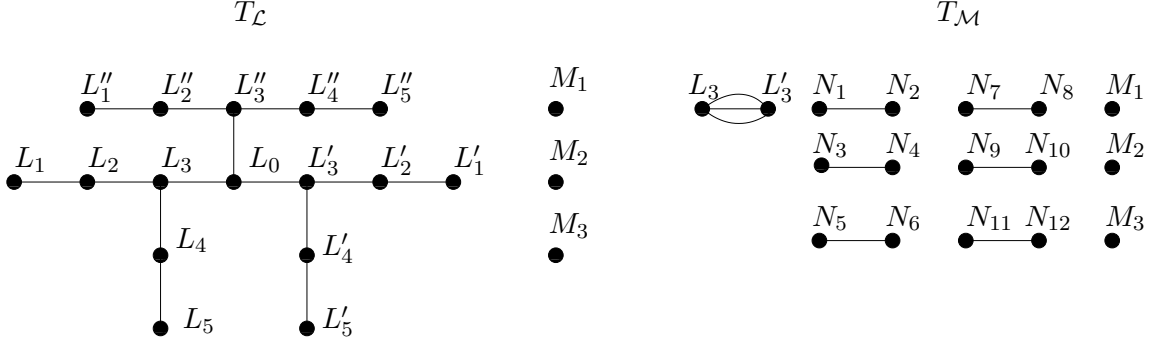$T_{\mathcal{L}}$ $\qquad\qquad\qquad\qquad\qquad\qquad$ $T_{\mathcal{M}}$

Fig. 3

$T_{\mathcal{M}}$. Moreover since $L_3 \cdot \mathcal{L} = 0$ the map ϕ is not ramified on L_3. In the same way since $L_3 \cdot \mathcal{M} = L_3' \cdot \mathcal{M} = 0$ the map ϕ is not ramified on L_3, L_3' and the two rational curves meet in three distinct points. Observe that the (-2)-curves of $Y'_{T \times T}$ remain (-2)-curves on $T_{\mathcal{L}}$ and on $T_{\mathcal{M}}$ and we have the configurations of (-2)-rational curves given in Figure 3.

These are 19 and 17 curves. The discriminants are the following:

	$d(L)$	$d(M)$	$d(N)$	d
$T_{\mathcal{L}}$	$-2^2 \cdot 3^3 \cdot 5$	-2^3		$2^5 \cdot 3^3 \cdot 5$
$T_{\mathcal{M}}$	-5	-2^3	3^6	$2^3 \cdot 3^6 \cdot 5$

Since it is not zero these classes are independent in the Neron-Severi group.

In the same way as before the curves $L_1, L_3, L_5, M_1, M_3, M_4, R_1, R_3$ disappear on $O_{\mathcal{L}}$ as the curves $L_1, L_3, L_5, L_1', L_3', L_5', M_1, M_2$ disappear on $O_{\mathcal{M}}$. Since $L_2 \cdot \mathcal{L} = L_4 \cdot \mathcal{L} = R_2 \cdot \mathcal{L} = 0$ and $L_2 \cdot \mathcal{M} = L_4 \cdot \mathcal{M} = L_2' \cdot \mathcal{M} = L_4' \cdot \mathcal{M} = 0$ the 2-cyclic coverings are not ramified there, and L_4, L_4' meet in two distinct points. Here the configurations of (-2)-rational curves are as in Figure 4.

These are 19 and 18 curves. The discriminants are the following:

	$d(L)$	$d(M)$	$d(N)$	$d(R)$	d
$O_{\mathcal{L}}$	$-2^2 \cdot 3 \cdot 7$	2^2	3^2	-2	$2^5 \cdot 3^3 \cdot 7$
$O_{\mathcal{M}}$	-7	2^4	3^2	2^4	$-2^8 \cdot 3^2 \cdot 7$

These are not zero hence the previous classes are independent in the Neron-Severi group. Finally observe that the canonical bundles on the surfaces are trivial, hence the surfaces are abelian or $K3$. Since in each case the rank of the Neron-Severi group is at least 17, it turns out that these are $K3$-surfaces and by construction are isomorphic to the surfaces announced in the statement. $\qquad\square$

Remark 7.1. *1. By taking the coverings $T_{\mathcal{L}'}$ resp. $T_{\mathcal{M}'}$ one gets "almost" the same surfaces as before, more precisely one has to conjugate the normal subgroup $T \times V$ with C, and to interchange (p_3, p_3) in $(TT)'$ with (p_3^2, p_3), getting the group $(TT)''$ (cf. Remark 3.1).*
2. By taking the covering $O_{\mathcal{L}'}$ one gets a K3-surface which is isomorphic with $Y_{T \times O}$. We do not discuss these surfaces separately.
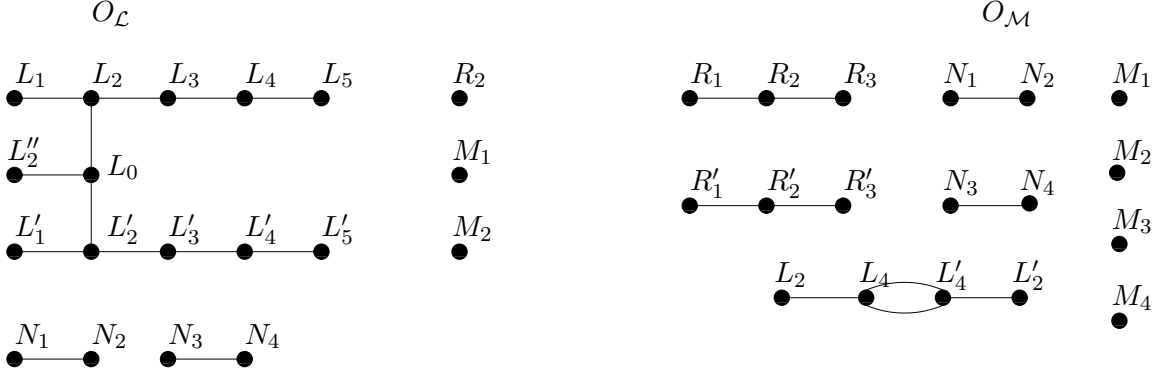
$O_{\mathcal{L}}$

$O_{\mathcal{M}}$

Fig. 4

7.2. **The special cases.** We use the same notation of [BS]. We denote by $X_{6,1}$, $X_{6,2}$, $X_{6,3}$, $X_{6,4}$ the four singular surfaces in the pencil X_λ^6 and by $X_{8,1}$, $X_{8,2}$, $X_{8,3}$, $X_{8,4}$ the four singular surfaces in the pencil X_λ^8. For the convenience of the reader we recall in Figure 5 also the graphs of the rational curves in the resolutions of the quotient, which come from singular points outside the quadric. In Figure 5 we do not draw separately the graphs of these curves for the surfaces $Y_{T\times T}^{(6,1)}$ and $Y_{T\times T}^{(6,4)}$ resp. of $Y_{T\times T}^{(6,2)}$ and $Y_{T\times T}^{(6,3)}$ since these looks equal, however one has to replace the curves N_1, N_2, N_3, N_4 by the curves N_5, N_6, N_7, N_8.

Now consider the projection π of Section 7. In these cases, the situation is a little more complicated. Now in the counterimage $\pi^{-1}(P)$ of some singular point P of X_λ/G coming from the A_1-singularities of X_λ we have singularities on X_λ/H too. In the following table we give the singularities in the quotient X_λ/G, $G = T \times T$ or $O \times O$ and the type and the number of singularities in the counterimage on X_λ/H:

	6,1	6,2	6,3	6,4		8,1	8,2	8,3	8,4
$T \times T$	E_6	A_5	A_5	E_6	$O \times O$	E_7	D_6	D_4	D_5
$T \times V$	D_4	A_1	A_1	D_4	$O \times T$	E_6	D_4	A_3	A_5
$(TT)'$	$3E_6$	$3A_5$	A_1	D_4	$(OO)''$	$2E_7$	A_7	A_3	$2D_5$
$V \times V$	$3D_4$	$3A_1$	$3A_1$	$3D_4$	$T \times T$	$2E_6$	A_3	A_1	$2A_5$

By resolving the quotients we get again a map like γ in Section 7. We can describe this map as there by using cyclic coverings. We distinguish two cases:

1. The case of $T \times T$. One constructs the 3-cyclic covering as in the general case by using the divisors $\mathcal{L}$, $\mathcal{L}'$, $\mathcal{M}$, $\mathcal{M}'$, these are in the case of the singular surfaces 3-divisible too. Then one blows down the (-1)-curves. The graphs of the (-2)-rational curves on the special surfaces and not coming from the lines of the base locus of X_λ^6 (these are the same as in the Figure 3) looks as in Figure 6 (cf. also [BS], Section 4.2). Observe that the graphs of the curves on $T_{\mathcal{L}}^{(6,1)}$ and on $T_{\mathcal{L}}^{(6,4)}$, resp. on $T_{\mathcal{L}}^{(6,2)}$ and on $T_{\mathcal{L}}^{(6,3)}$ are the same. This together with the (-2)-curves coming from the base locus of the pencil X_λ^6 gives the following numbers of independent curves in the Neron-Severi group:

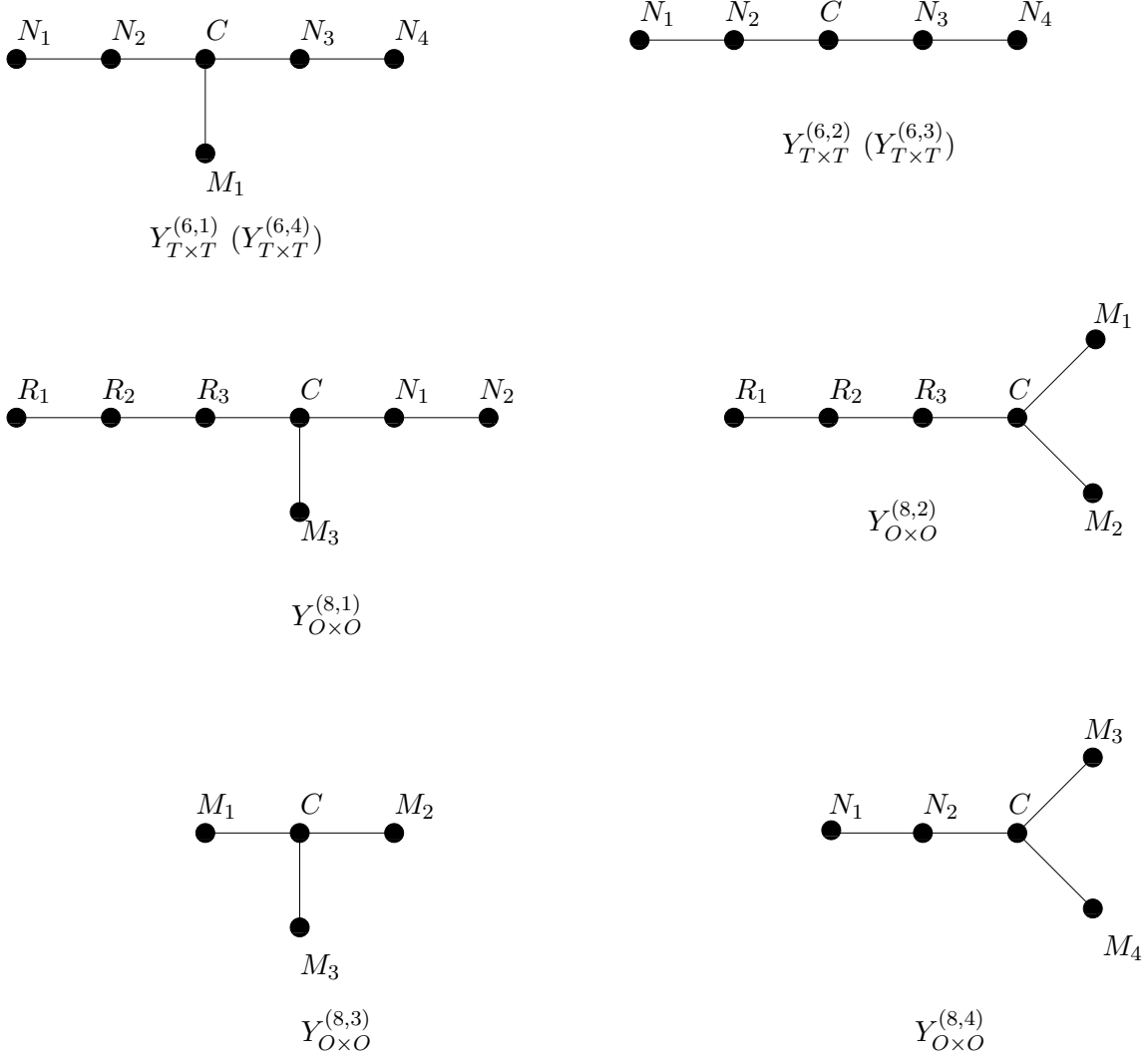| $\mathcal{L}$	6,1	6,2	6,3	6,4		$\mathcal{M}$	6,1	6,2	6,3	6,4
	20	20	20	20	,		20	20	18	18

Fig. 5

Observe that by taking the divisor $\mathcal{L}'$ one obtains the same graphs, by taking the divisor $\mathcal{M}'$ one finds the following isomorphisms of graphs and, by construction, of surfaces:

$$T_{\mathcal{M}'}^{(6,1)} \cong T_{\mathcal{M}}^{(6,4)}, \; T_{\mathcal{M}'}^{(6,2)} \cong T_{\mathcal{M}}^{(6,3)}, \; T_{\mathcal{M}'}^{(6,3)} \cong T_{\mathcal{M}}^{(6,2)}, \; T_{\mathcal{M}'}^{(6,4)} \cong T_{\mathcal{M}}^{(6,1)}.$$

In fact one sees that by taking $\mathcal{L}$ or $\mathcal{L}'$ one obtains the special $K3$-surfaces in the family $Y_{\lambda,T\times V}$ (up to conjugate the group $T \times V$ with the matrix C), by taking $\mathcal{M}$ or $\mathcal{M}'$ one obtains the special $K3$-surfaces in the covering $Y_{\lambda,(TT)'}$ or $Y_{\lambda,(TT)''}$.

2. *The case of $O \times O$.* We take now the divisors $\mathcal{L}, \mathcal{L}', \mathcal{M}$ and do 2-cyclic coverings, then blow down the (-1)-curves. The graphs of (-2)-rational curves on the special surfaces not coming from the lines of the base locus of the pencil X_λ^8 (these looks like in Figure 4) are as in Figure 7.
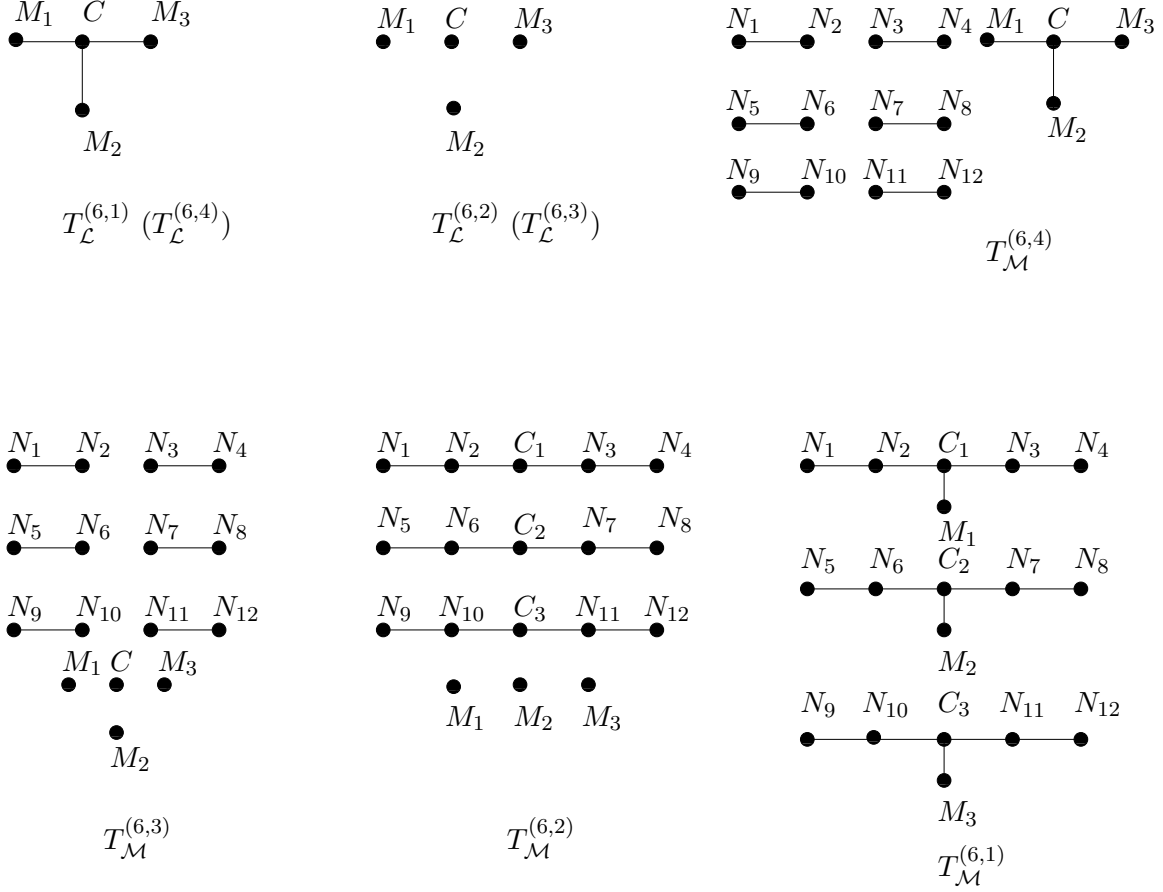
Fig. 6

These together with the (-2)-curves coming from the base locus of the pencil X_λ^8 give the following numbers of independent curves in the Neron-Severi group:

$\mathcal{L}$	8,1	8,2	8,3	8,4		$\mathcal{M}$	8,1	8,2	8,3	8,4
	20	20	20	20			20	19	19	20

Similarly to the general case, by taking $\mathcal{L}$ or $\mathcal{L}'$ we find the singular surfaces in the family $Y_{\lambda, O \times T}$ (up to conjugate $O \times T$ with the matrix C) and by taking $\mathcal{M}$ we find the singular surfaces in the family $Y_{\lambda, (OO)''}$.

In both the cases 1 and 2 the discriminants of the lattices spanned by the previous curves are those given in Section 5.

8. Picard-lattices

We compute the Picard-lattices of the general K3-surface in the families $T_\mathcal{L}$, $O_\mathcal{L}$ and of the special surfaces with $\rho = 20$ in each pencil. First we recall some facts. Denote by W the lattice spanned by the curves of Proposition 7.2. If W is not the total Picard-lattice, which we call NS there is an integral lattice W' s.t. $W \subset W' \subset NS$ with $p := [W' : W]$ a prime number. Denote by $d(W)$, $d(W')$ the discriminant of the lattices W, W'. Since $[W' : W]^2 = d(W) \cdot d(W')^{-1}$ (cf. [BPV] Lemma 2.1, p. 12) we find that p^2 divides the discriminant of W. Denote by $(W^\vee/W)^p$ the p-subgroup of $(W^\vee/W) \subset NS^\vee/NS$ and denote

N_1 N_2 C N_3 N_4 　　R_1 R_2 R_3 C_1 N_1 N_2 　M_3 　　M_1 C M_2

R_2 　　　　　　　　　M_1 　　　　　　　　　　R_2

M_1 　　M_2 　　R'_1 R'_2 R'_3 C_2 N_3 N_4 　M_4 　　N_1 N_2

$O^{(8,1)}_{\mathcal{L}}$ 　　　　　　　　　　M_2 　　　　　　　　N_3 N_4

　　　　　　　　　　$O^{(8,1)}_{\mathcal{M}}$ 　　　　　　　　$O^{(8,2)}_{\mathcal{L}}$

R_1 R_2 R_3 C R'_1 R'_2 R'_3 　M_1 C M_2 R_2 　R_1 R_2 R_3 N_1 N_2

N_1 N_2 M_1 M_2 　　　　　　N_1 N_2 N_3 N_4 　R'_1 R'_2 R'_3 N_3 N_4

N_3 N_4 M_3 M_4 　　　　　　　　　　　　　　M_1 C M_2 M_3 M_4

　　　　　　　　　　　　　　$O^{(8,3)}_{\mathcal{L}}$

$O^{(8,2)}_{\mathcal{M}}$ 　　　　　　　　　　　　　　$O^{(8,3)}_{\mathcal{M}}$

N_1 N_2 C N_3 N_4 　　　　R_1 R_2 R_3 N_1 N_2 C_1 M_1

M_1 M_2 R_2 　　　　　　　　　　　　　　　　M_2

$O^{(8,4)}_{\mathcal{L}}$ 　　　　　　R'_1 R'_2 R'_3 N_3 N_4 C_2 M_3

　　　　　　　　　　$O^{(8,4)}_{\mathcal{M}}$ 　　M_4

Fig. 7

by T the transcendental lattice orthogonal to the Picard-lattice. Since the discriminant groups $T^{\vee}/T$ and $NS^{\vee}/NS$ are isomorphic (cf. e.g. [BPV] p. 13 Lemma 2.5), they have the same rank which is $\leq \mathrm{rk}(T)$. It follows that also $\mathrm{rk}(W^{\vee}/W)^p \leq \mathrm{rk}(T)$.

8.1. The general case.

Proposition 8.1. *1. The class:*

$$\bar{L}' := L_1 - L_2 + L_4 - L_5 + L'_1 - L'_2 + L'_4 - L'_5 + L''_1 - L''_2 + L''_4 - L''_5$$

is 3-divisible in $NS(T_{\mathcal{L}})$ and the classes:

$$h_1 := L_1 + L_3 + L_5 + L'_1 + L'_3 + L'_5 + M_1 + M_2,$$
$$h_2 := L_1 + L_3 + L_5 + L''_1 + L''_3 + L''_5 + M_1 + M_3$$

are 2-divisible in $NS(T_{\mathcal{L}})$. These together with the 19 curves of Proposition 7.2 span a 19-dimensional lattice with discriminant $2 \cdot 3 \cdot 5$.

2. The class:

$$\bar{L}' := L_1 + L_3 + L_5 + L_1' + L_3' + L_5' + M_1 + M_2$$

is 2-divisible in $NS(O_{\mathcal{L}})$ and the class:

$$k_1 := L_1 - L_2 + L_4 - L_5 - L_1' + L_2' - L_4' + L_5' + N_1 - N_2 + N_3 - N_4$$

is 3-divisible. These together with the 19 curves of Proposition 7.2 span a 19-dimensional lattice with discriminant $2^3 \cdot 3 \cdot 7$.

Proof. 1. The discriminant of the lattice generated by the 19 curves is $2^5 \cdot 3^3 \cdot 5$ hence we can have 2-divisible classes or 3-divisible classes. The divisor $\bar{L}'$ is 3-divisible since it is the pull back of the divisor $\mathcal{L}'$ on $Y_{T \times T}$ which is 3-divisible too. And we cannot have more 3-divisible classes. If there are no 2-divisible classes then the group $(W^\vee/W)^2$ would contain the classes $M_1/2$, $M_2/2$, $M_3/2$, $(L_1 + L_3 + L_5 + L_1' + L_3' + L_5')/2$, $(L_1 + L_3 + L_5 + L_1'' + L_3'' + L_5'')/2$, $(L_1' + L_3' + L_5' + L_1'' + L_3'' + L_5'')/2$ which are independent classes with respect the intersection form. Since the rank of $(W^\vee/W)^2$ is less or equal as the rank of $T^\vee/T$ which is at most three, it can not happen that we find five classes as before. Hence some combination of them must be contained in the Neron-Severi group. So we have

$$\frac{1}{2}(\lambda(L_1 + L_3 + L_5) + \lambda'(L_1' + L_3' + L_5') + \lambda''(L_1'' + L_2'' + L_3'') + \mu_1 M_1 + \mu_2 M_2 + \mu_3 M_3) \in NS$$

for some parameters $\lambda, \lambda', \lambda'', \mu_1, \mu_2, \mu_3 \in \mathbb{Z}_2$.

By Nikulin [N] such a 2-divisible set contains 8 curves. So putting $\lambda'' = 0$ and $\mu_3 = 0$ we get the divisor h_1, putting $\lambda' = 0$ and $\mu_2 = 0$ we get the divisor h_2. The discriminant of the lattice W together with these three classes now change into $2 \cdot 3 \cdot 5$, hence we cannot have more torsion classes.

2. Again the class $\bar{L}'$ is the pull back of the class $\mathcal{M}'$ on $Y_{O \times O}$ hence it is 2-divisible. If there are no 3-divisible classes then the group $(W^\vee/W)^3$ would contain the classes $N_1 - N_2/3$, $N_3 - N_4/3$ and $(L_1 - L_2 + L_4 - L_5 + L_1' - L_2' + L_4' - L_5')/3$ which are independent. We have these three independent classes also on the special surfaces $O_{\mathcal{L}}^{(8,2)}$ and on $O_{\mathcal{L}}^{(8,3)}$. Hence we would have also there $\mathrm{rk}(W^\vee/W)^3 \geq 3$. This is not possible in fact on these surfaces we have $\mathrm{rk}(W) = 20$ which implies $\mathrm{rk}(W^\vee/W)^3 \leq 2$. This means that the three classes fit together giving a 3-divisible class in $NS(O_{\mathcal{L}}^{(8,2)})$ and $NS(O_{\mathcal{L}}^{(8,3)})$. Now with the help of a blow-up it is possible to resolve the singularities of the total space $\mathbb{P}_3/O \times T$ of the family in correspondence of the singular fibers $O_{\mathcal{L}}^{(8,2)}$ and $O_{\mathcal{L}}^{(8,3)}$, obtaining a smooth threefold there, and where the two singular surfaces are replaced by surfaces with only A_i-singularities. Then by using a result of Briskorn, cf. [Br], one can replace these two fibers by smooth fibers, the total space remains smooth. Hence we see that the previous class which we have on the special fibers also exists in the Neron-Severi group of the generic fiber $O_{\mathcal{L}}$.

$\square$

Remark 8.1. *In the proof of Proposition 8.1, 2, we need a special resolution of the total space which induces a resolution of two special fibers. To do this we make a simultaneous blow-up of the threefold singularities and of the surface singularities. In fact we are able to do this in the case that the singularities of the surface is an A_i or a D_n, n even, singularity. We do not know how to find such blow up if the fibers have D_n, n odd, singularities, and E_6, E_7 or E_8 singularities. One method to find the necessaries resolutions also in these cases is*

to use the famous McKay-correspondence, between group representations and resolutions of singularities (cf. [IN]). This way works and it is the result of a forthcoming paper [LS] of M. Lehn and the author.

8.2. The special cases.

Proposition 8.2. *1. The Picard-lattice of the special surfaces in $T_{\mathcal{L}}$ and $O_{\mathcal{L}}$ is generated in all the cases but $O_{\mathcal{L}}^{(8,4)}$ by the curves of Proposition 7.2 and by the 2-divisible classes of Proposition 8.1. In the case of $O_{\mathcal{L}^{(8,4)}}$ the class:*

$$L_1 + L_3 + L_5 + N_1 + C + N_4 + R_2 + M_1$$

is 2-divisible too.
2. In the case of $T_{\mathcal{M}}^{(6,1)}$ and of $T_{\mathcal{M}}^{(6,2)}$ the class:

$$\bar{L} := N_1 - N_2 + N_3 - N_4 + N_5 - N_6 + N_7 - N_8 + N_9 - N_{10} + N_{11} - N_{12}$$

is 3-divisible. Moreover in the case of $T_{\mathcal{M}}^{(6,2)}$ the classes:

$$N_1 + C_1 + N_4 + N_5 + C_2 + N_8 + M_1 + M_2,$$
$$N_1 + C_1 + N_4 + N_9 + C_3 + N_{12} + M_1 + M_3$$

are 2-divisible. These together with the 20 curves of Section 7.2 span a 20-dimensional lattice.
3. In the case of $O_{\mathcal{M}}^{(8,1)}$ and $O_{\mathcal{M}}^{(8,4)}$, the class:

$$\bar{L} := M_1 + M_2 + M_3 + M_4 + R_1 + R_3 + R_1' + R_3'$$

is 2-divisible and in the case of $O_{\mathcal{M}}^{(8,4)}$ the class:

$$W := R_1 + 2R_2 + 3R_3 + R_1' + 2R_2' + 3R_3' + 2N_1 + 2C_1 + 3M_1 + M_2 + 2N_3 + 2C_2 + 3M_3 + M_4$$

is 4-divisible.
This together with the 20 curves of Section 7.2 spans a 20-dimensional lattice.
The ranks and the discriminants of the Picard-lattices are:

	$T_{\mathcal{L}}$				$T_{\mathcal{M}}$	
	$6,1$	$6,2$	$6,3$	$6,4$	$6,1$	$6,2$
rank	20	20	20	20	20	20
discriminant	$-3 \cdot 5$	$-2^2 \cdot 3 \cdot 5$	$-2^2 \cdot 3 \cdot 5$	$-3 \cdot 5$	$-3 \cdot 5$	$-2^2 \cdot 3 \cdot 5$

	$O_{\mathcal{L}}$				$O_{\mathcal{M}}$	
	$8,1$	$8,2$	$8,3$	$8,4$	$8,1$	$8,4$
rank	20	20	20	20	20	20
discriminant	$-2^2 \cdot 7$	$-2^2 \cdot 3 \cdot 7$	$-2^3 \cdot 3 \cdot 7$	$-2^2 \cdot 7$	$-2^2 \cdot 7$	$-2^4 \cdot 7$

Proof. (cf. also [BS] Theorem 6.2) 1. In all the special cases of the pencils $T_{\mathcal{L}}$ and $O_{\mathcal{L}}$ we do not have enough rational (-2)-curves satisfying the conditions of [N] and [T] to obtain more 2-divisible or 3-divisible classes.
2. The class $\bar{L}$ comes from the 3-divisible class $\mathcal{L}$ on $Y_{T \times T}$ hence it is 3-divisible too. One proves as in Proposition 8.1 that in the case of $T_{\mathcal{M}}^{(6,2)}$ we have two more divisible classes.
3. The class $\bar{L}$ comes from the class $\mathcal{L}$ on $Y_{O \times O}$, hence it is 2-divisible on $O_{\mathcal{M}}^{(8,1)}$ and $O_{\mathcal{M}}^{(8,4)}$.
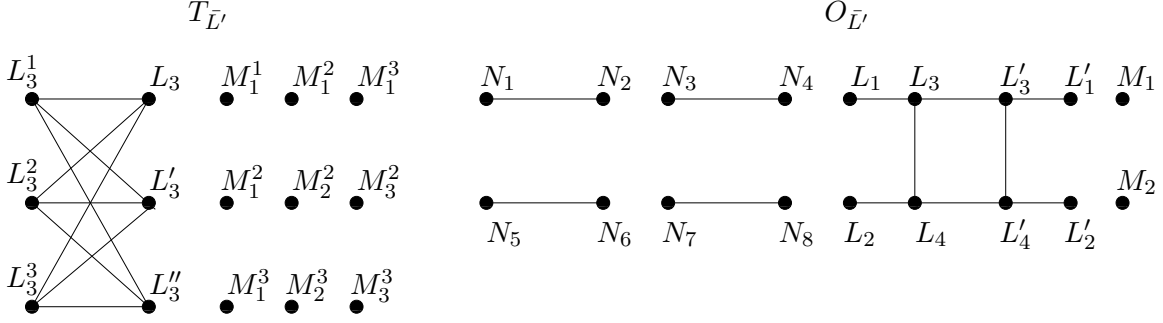
$T_{\bar{L}'}$ $O_{\bar{L}'}$

L_3^1 L_3 M_1^1 M_1^2 M_1^3 N_1 N_2 N_3 N_4 L_1 L_3 L_3' L_1' M_1

L_3^2 L_3' M_1^2 M_2^2 M_3^2 M_2

L_3^3 L_3'' M_1^3 M_2^3 M_3^3 N_5 N_6 N_7 N_8 L_2 L_4 L_4' L_2'

Fig. 8

Consider the classes:

$$v_1 := R_1 + 2R_2 + 3R_3,$$
$$v_2 := R_1' + 2R_2' + 3R_3',$$
$$v_3 := 2N_1 + 2C_1 + 3M_1 + M_2,$$
$$v_4 := 2N_3 + 2C_2 + 3M_3 + M_4$$

on $O_{\mathcal{M}}^{(8,4)}$. Since the rank of NS is 20 the classes $v_i/4$, $i = 1, 2, 3, 4$ cannot be independent in $NS^{\vee}/NS$, hence we must find coefficients $\alpha_i \in \mathbb{Z}_4$ with:

$$W := 1/4(\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + \alpha_4 v_4) \in NS.$$

Observe that then $2W \in NS$ too and so the class:

$$1/2(\alpha_1(R_1 + R_3) + \alpha_2(R_2' + R_3') + \alpha_3(M_1 + M_2) + \alpha_4(M_3 + M_4))$$

must be in NS too. By [N] this must contains eight or 16 curves, hence $\alpha_i = 1$ (mod $2\mathbb{Z}$) and after perhaps interchanging v_i with $-v_i$, we may assume $\alpha_i = 1$ (mod $4\mathbb{Z}$). In this way we find the class of the statement. Finally by the same argumentation of 1 and 2 we cannot have more 3-divisible classes or 2-divisible classes. $\square$

9. More cyclic coverings

9.1. **The general case.** As in Section 7 we can construct the 3-cyclic covering of $T_{\mathcal{L}}$, $T_{\mathcal{M}}$ by using the 3-divisible classes $\bar{L}'$ and $\bar{L}$ and the 2-cyclic coverings of $O_{\mathcal{L}}$, $O_{\mathcal{M}}$ by using the 2-divisible divisors $\bar{L}'$, $\bar{L}$. As before we consider first the non-special surfaces. Call the coverings $T_{\bar{L}'}$, $T_{\bar{L}}$, resp. $O_{\bar{L}'}$, $O_{\bar{L}}$. Then:

Proposition 9.1. *The surfaces $T_{\bar{L}'}$ and $T_{\bar{L}}$ are isomorphic and the surfaces $O_{\bar{L}'}$ and $O_{\bar{L}}$ too. These are K3-surfaces which contains 15 rational curves coming from the 19 curves of $T_{\mathcal{L}}$, and 18 rational curves coming from the 19 curves of $O_{\mathcal{L}}$, moreover are isomorphic with $Y_{V \times V}$ and with $Y_{T \times T}$.*

Proof. The irreducible curves contained on the divisor $\bar{L}'$ disappear on $T_{\bar{L}'}$ and the same happens on the surface $O_{\bar{L}'}$. On these surfaces we have the configuration of 15, resp. 18 (-2)-rational curves given in Figure 8.
Similarly to Proposition 7.2 we write down the discriminants:

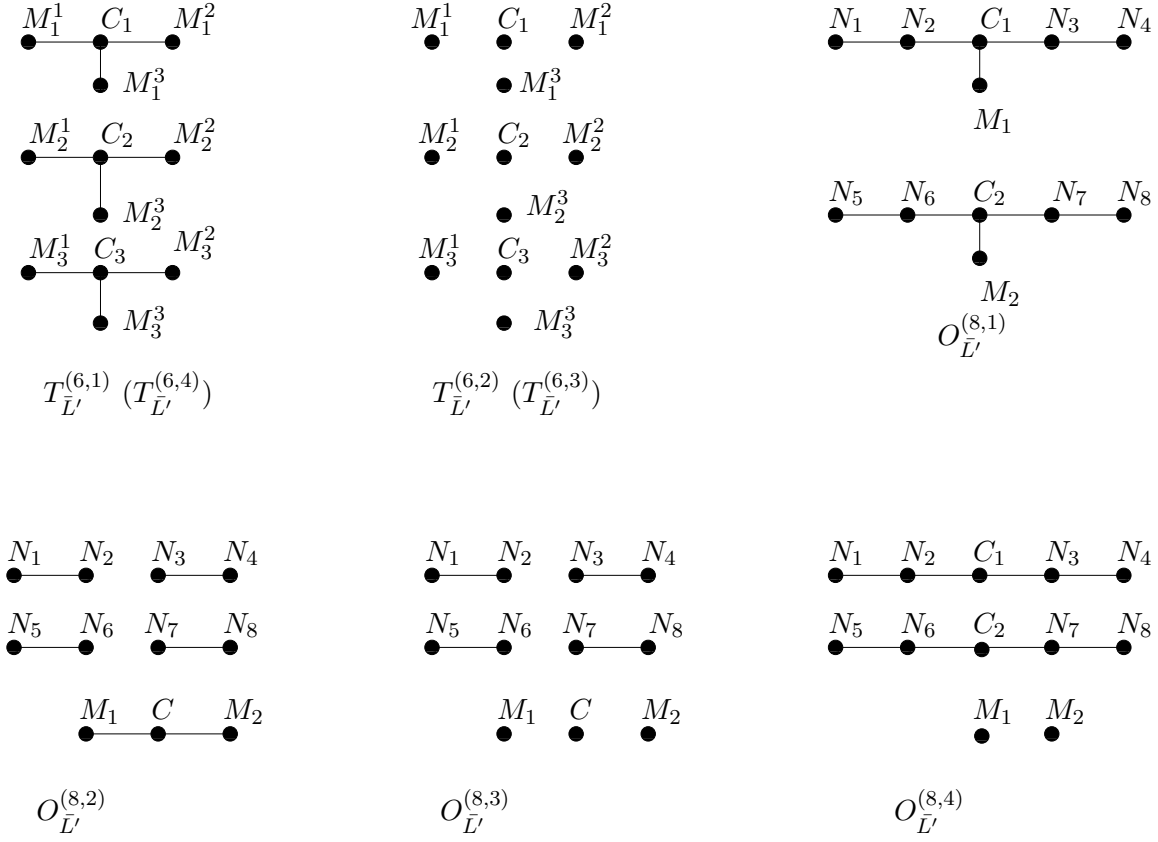|	$d(L)$	$d(M)$	$d(N)$	d
$T_{\bar{L}'}$	$-2^4 \cdot 5$	-2^9		$2^{13} \cdot 5$
$O_{\bar{L}'}$	$-3^2 \cdot 7$	2^2	3^4	$-2^2 \cdot 3^6 \cdot 7$

Fig. 9

The surfaces are K3 and are isomorphic with $Y_{V \times V}$ and with $Y_{T \times T}$. Namely observe that $V \times V$ is a subgroup of index 3 of the groups $T \times V$ and of $(TT)'$, hence we have a diagram as those given in Figure 2. Similarly $T \times T$ is an index 2 subgroup of $O \times T$ and of $(OO)''$ so as before we have the diagram of Figure 2. □

9.2. **The special cases.** We construct 3-cyclic coverings of the special $T_{\mathcal{L}}$ and $T_{\mathcal{M}}$ by using the 3-divisible classes $\bar{L}'$ and $\bar{L}$ and the 2-cyclic covering of the special $O_{\mathcal{L}}$ and $O_{\mathcal{M}}$ by using the 2-divisible classes $\bar{L}'$, $\bar{L}$ (in the same way as in Section 7). Other than the curves coming from the base locus of X_λ^6 and X_λ^8 we have the configurations of rational curves given in Figure 9. Observe that as in the general case the surfaces $T_{\bar{L}'}$ are isomorphic to the surfaces $T_{\bar{L}}$ and the surfaces $O_{\bar{L}'}$ are isomorphic to the $O_{\bar{L}}$.

The surfaces which we obtain are exactly the special K3-surfaces in the families $Y_{\lambda, V \times V}$ and $Y_{\lambda, T \times T}$. The number of independent rational curves in the Neron-Severi group is:

$\bar{L}(\bar{L}')$	6, 1	6, 2	6, 3	6, 4	8, 1	8, 2	8, 3	8, 4
	18	18	18	18	20	19	19	20

This gives a lower bound for the Picard-number, the discriminants of the lattices generated by these curves are given in Section 5.

9.3. **The Picard-Lattices.** In this Section we identify the Picard-lattices of the surfaces $O_{\bar{L}'}^{(8,1)}$ and $O_{\bar{L}'}^{(8,4)}$ which have $\rho = 20$. The proof of the following Proposition is left to the reader since it is very similar to the proof of Propositions 8.1 and 8.2.

Proposition 9.2. *The classes:*

$$k_1' = L_2 - L_4 + L_3' - L_1' + N_1 - N_2 + N_3 - N_4 + N_5 - N_6 + N_7 - N_8,$$
$$k_1'' = L_2' - L_4' + L_3 - L_1 + N_1 - N_2 - N_3 + N_4 + N_5 - N_6 - N_7 + N_8$$

are 3-divisible in $NS(O_{\bar{L}'}^{(8,1)})$, $NS(O_{\bar{L}'}^{(8,4)})$. *Moreover in the case of* $O_{\bar{L}'}^{(8,4)}$ *the class:*

$$\kappa = N_1 + C_1 + N_4 + N_5 + C_2 + N_8 + M_1 + M_2$$

is 2-divisible. These classes together with the (-2)-*curves of Section 9.2 span lattices of the following ranks and discriminants:*

	$8,1$	$8,4$
rank	20	20
discriminant	-7	$-2^2 \cdot 7$

Remark 9.1. *Observe that the surfaces* $Y_{\lambda,V\times V}$ *and* $Y_{\lambda,T\times T}$ *are for each* $\lambda \in \mathbb{P}_1$ *the fiber products*

$$Y_{\lambda,T\times V} \times Y_{\lambda,(TT)'} \ \ over \ Y_{\lambda,T\times T}^6$$
$$Y_{\lambda,O\times T} \times Y_{\lambda,(OO)''} \ \ over \ Y_{\lambda,O\times O},$$

(to avoid confusion we denote here by $Y_{T\times T}^6$ *the minimal resolution of* $X_\lambda^6/T \times T$).

10. Final Remarks

1. In the Sections 8 and 9 we identify explicitly the Picard-lattice of some K3-surfaces. It is our next aim to compute the transcendental lattices orthogonal to the Picard-lattices to classify the K3-surfaces. In particular by a result of Shioda and Inose, cf. [SI], K3-surfaces with $\rho = 20$ are classified by means of their transcendental lattice.

2. By a result of Morrison, cf. [Mo], each K3-surface with $\rho = 19$ or 20 admits a so called Shioda-Inose structure. This means that there is a Nikulin-involution, an involution with eight isolated fix-points and the quotient is birational to a Kummer-surface. It would be desirable to have an explicit description of this structure for our surfaces.

3. The quotients 3-folds $\mathbb{P}_3/G$, $G = T \times T, O \times O, I \times I$ or a subgroup described in Section 2 seem to be Fano. We did not describe these spaces here. It would be interesting to have a global resolution of these spaces and to see our $K3$-surfaces as pencils on the smooth 3-folds. A first result which is helpful in finding such resolutions is contained in [LS].

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